dubious calculatory diversions and maintains a direct reference to population figures. It allocates one non-base seat for every 856 500 citizens or part thereof, except when the minimum restrictions warrant more seats or the maximum cap imposes fewer seats. In Table 12.5 all exceptions are marked with a dot(•). The limitation of losses may find its constitutional justification in the pragmatic principle of electoral continuity that exempts the legislator from amending the electoral system too abruptly when a gentle transition is also feasible.

12.10. JAGIELLONIAN COMPROMISE

The *Jagiellonian Compromise* proposes a qualified majority voting system for the European Council. The council is the decision-making body of the Member States' governments. Hence the Jagiellonian Compromise does not belong to the systems for representing people that are in the focus of this book. Yet the procedure pays due attention to the status of the Union's citizens. In this sense the Jagiellonian Compromise is in agreement with the Cambridge Compromise, its later namesake. In view of the philosophical affinity we take the space to briefly outline its essence.

In a QMV system, every Member State is equipped with a voting weight. A group of Member States qualifies as a majority when their cumulative voting weights meet or exceed a preordained quota.

The Jagiellonian Compromise derives its voting weights and quota directly from the Member States' population figures, and it does so in a remarkably transparent way. The *voting weight* of Member State i simply is the square root of its QMV-population, $\sqrt{p_i}$. The *quota* q is calculated from the QMV-populations by averaging the sum of the square roots and the square root of the sum,

$$q := \frac{1}{2}\left(\sum_{i\leq 28}\sqrt{p_i} + \sqrt{\sum_{i\leq 28} p_i}\right).$$

Table 12.6 exhibits weights and quota for the year 2013. For the sake of simplicity all quantities are commercially rounded. The quota 60 723 amounts to 61.4 percent of the sum of the voting weights.

In the past the council and its precursors agreed on weights and quota by negotiation. The negotiated systems cast doubt on the expertise of European diplomats and their advisers. In the European Economic Community 1958–1972, Germany, France and Italy each had weight 4, the Netherlands and Belgium 2 each, and Luxembourg one. The quota was set to be 12. This system gave Luxembourg no decision power whatsoever. The other states commanded twelve votes or more, or ten votes or less. Luxembourg was never decisive to reach the quota, or to fail it. In 1973 Luxembourg was assigned two votes, and Ireland and Denmark three each. For the ten Member States 1981–1985 the quota was set at 45. Among the 1024 possible voting profiles Luxembourg was decisive as often as were Ireland and Denmark. The three states had the same decision power despite of the ostensible distinctness of their voting weights. Diplomatic good-will cannot substitute for factual expertise.

EuropeanCouncil2013	QMV-Population	Voting Weight	Power Share
Germany	81 843 700	9 047	9.15
France	65 397 900	8 087	8.18
United Kingdom	62 989 600	7 937	8.02
Italy	60 820 800	7 799	7.89
Spain	46 196 300	6 797	6.87
Poland	38 538 400	6 208	6.28
Romania	21 355 800	4 621	4.67
Netherlands	16 730 300	4 090	4.14
Greece	11 290 900	3 360	3.40
Belgium	11 041 300	3 323	3.36
Portugal	10 541 800	3 247	3.28
Czech Republic	10 505 400	3 241	3.28
Hungary	9 957 700	3 156	3.19
Sweden	9 482 900	3 079	3.11
Austria	8 443 000	2 906	2.94
Bulgaria	7 327 200	2 707	2.74
Denmark	5 580 500	2 362	2.39
Slovakia	5 404 300	2 325	2.35
Finland	5 401 300	2 324	2.35
Ireland	4 582 800	2 141	2.16
Croatia	4 398 150	2 097	2.12
Lithuania	3 007 800	1 734	1.75
Slovenia	2 055 500	1 434	1.45
Latvia	2 041 800	1 429	1.44
Estonia	1 339 700	1 157	1.17
Cyprus	862 000	928	0.94
Luxembourg	524 900	724	0.73
Malta	416 100	645	0.65
Sum	508 077 850	98 905	100.00
Quota		60 723	61.40

TABLE 12.6 *Jagiellonian Compromise proposal for the European Council.* A Member State's voting weight is the square root of its QMV-population. A group of Member States qualifies as a majority when their cumulative weights meet or exceed the quota, 60 723. The quota is the average of the sum of the voting weights (98 905) and the square-root of the population total ($\sqrt{508\,077\,850}$). The system gives all Union citizens the same power to contribute indirectly via their governments.

The number of voting profiles in which Member State i turns decisive by passing from Yea to Nay or from Nay to Yea provides a meaningful measure for the state's decision power. The 28 Member States of the current Union allow 268 million profiles of dividing into Yea and Nay camps. Diplomatic insight no longer suffices to count how often a state is tipping the scales. Elaborate procedures are needed to carry out the counting, in general. Specifically, the Jagiellonian Compromise comes with a surprise: A state's share of power is proportional to its voting weight! The proof of the statement is difficult, but the statement itself makes life easy. All that is needed to determine the power distribution among the 28 Member States is to normalize the voting weights. The resulting *power share* $\beta_i := \sqrt{p_i}/\sum_{j\leq 28}\sqrt{p_j}$ of Member State i is shown in the last column of Table 12.6 (in percent). The power share β_i is also called its *normalized Banzhaf index* whence the use of the letter β.

The crux of the Jagiellonian Compromise is the quota formula. The formula is due to *Słomczyński / Życzkowski* (2006), two members of the Jagiellonian University Kraków, thus explaining the attribute "Jagiellonian". The system is a veritable "compromise" because it gently mediates between the decision power biases and other characteristics of several QMV systems currently in use or under discussion. The seminal monograph of *Felsenthal / Machover* (1998) teaches how to comprehensively analyze QMV systems. In particular it shows how to appraise a QMV system from the citizens' viewpoint.

The basic assumption is that decision-making is a repetitive business. Therefore, system indices must be evaluated by their likely values, not by a single realization. The citizens' influence whether a proposal is carried is modeled by a thought experiment in an idealized democracy. First a popular vote is taken, and then the government executes the majority's will. From an *a priori* viewpoint it appears constitutionally compelling to assume that citizens cast their votes independently of each other. The critical proposals are those that are supported or dismissed with the same likelihood, one-half. Of course the influence of a single citizen tends to zero, particularly in a state i where the population p_i is large. On the other hand a large population of voters gives rise to an almost infinite set of nip-and-tuck races where the last vote becomes decisive. Since a limit of the form $0 \times \infty$ has no meaning, a more sensitive analysis is called for. As the population grows, $p_i \to \infty$, an individual citizen's decision power decreases as $1/\sqrt{p_i}$. This is a consequence of the Central Limit Theorem. Let $X_n = 1$ indicate a Yea of citizen n, and $X_n = 0$ a Nay. The outcome is determined by the Yea total, $\sum_{n \le p_i} X_n$, a divergent sum. It needs to be scaled by $1/\sqrt{p_i}$ to be stabilized; this is the point where the square root makes its appearance. It can then be shown that if Member State i has a decision power β_i on the government level, the system conveys indirect power $\beta_i/\sqrt{p_i}$ to every citizen.

The discussion of the Jagiellonian Compromise now is quickly concluded. As mentioned above the government of Member State i has direct decision power $\beta_i = \sqrt{p_i}/\sum_{j \le 28} \sqrt{p_j}$. Since $\beta_i/\sqrt{p_i} = 1/\sum_{j \le 28} \sqrt{p_j}$ is the same constant for all Member States i, the citizens of all Member States share the same indirect decision power. This is the second surprise that comes with the Jagiellonian Compromise: All Union citizens have the same indirect decision power!

The final conclusion is exceedingly gratifying. The Jagiellonian Compromise and the Cambridge Compromise both boost the democratic foundations of the Union. The Jagiellonian Compromise guarantees that all Union citizens participate with provable equality in the decision-making processes of the European Council, even though participation is indirect through their governments. The Cambridge Compromise implements a dual concept of equal representation in the EP by achieving equality of the Member States' citizenries as well as equality of the Union's citizens. The next chapter turns to a multi-goal problem of a different type, the reconciliation of proportional representation and the election of persons.

Friedrich Pukelsheim

Proportional Representation

Apportionment Methods and Their Applications

Springer

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With a Foreword by Andrew Duff MEP

Springer

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The just, then, is a species of the proportionate .... For proportion is equality of ratios, and involves four terms at least ...; and the just, too, involves at least four terms, and the ratio between one pair is the same as that between the other pair; for there is a similar distinction between the persons and between the things. As the term A, then, is to B, so will C be to D, and therefore, *alternando*, as A is to C, B will be to D. ...

This, then, is what the just is—the proportional; the unjust is what violates the proportion. Hence one term becomes too great, the other too small, as indeed happens in practice; for the man who acts unjustly has too much, and the man who is unjustly treated too little, of what is good. ...

This, then, is one species of the just.

Aristotle, Nichomachean Ethics, Book V, Chapter 3.
Translated and Introduced by Sir *David Ross*, 1953.
The World's Classics 546, Oxford University Press.

Foreword

The virtue of parliamentary democracy rests on the representative capability of its institutions. Even mature democratic states cannot take the strength of its representative institutions for granted. Newer democracies seek practicable ways and means on which to build lasting structures of governance which will command the affinity of the people they are set up to serve. The debate about the structural reform of parliamentary democracies is never far away. Nor should it be. The powers and composition of parliamentary chambers, their rules and working methods, the organisation and direction of the political parties which compete for votes and seats, the electoral systems (who to register, how to vote, how to count), the size and shape of constituencies—all these and more are rightly subject to continual appraisal and are liable to be reformed.

Electoral reform is a delicate business: handled well, it can be the basis on which new liberal democracies spread their wings; it can refresh the old, tired democracies. Handled badly, electoral reform can distort the people's will, entrench the abuse of power and sow the seeds of destruction of liberty. Electoral systems are central to the debate in the emerging democracies, and the relatively new practice of election observation by third parties highlights the need for elections to be run not only fairly but also transparently. Voting and counting should be simple, comprehensible and open to scrutiny—qualities which are too often lacking even in old established democracies.

Electoral reform is also very difficult to achieve. Those who must legislate for it are those very same people who have a vested interest in the status quo. That *Turkeys don't vote for Christmas* is amply demonstrated in the United Kingdom, where reform of the House of Lords has been a lost cause for over a century. Advocates of reform need to stack up their arguments well, be persistent and enjoy long lives.

Friedrich Pukelsheim has written a definitive work on electoral reform. He takes as his starting point the simple premise that seats won in a parliamentary chamber must represent as closely as possible the balance of the votes cast in the ballot box. Rigorous in his methodology, the author knows that there is no single perfect electoral system: indeed, in their quirky details every system affects the exact outcome of an election. We are fortunate indeed that this Professor of Mathematics is a profound democrat. He ably brings to the service of politicians the science of the mathematician.

Dr Pukelsheim was an indispensable participant at the meeting in Cambridge in 2011, chaired by Geoffrey Grimmett, which devised "CamCom"—the best consensual solution to the problem of how to apportion seats in the European Parliament. As the Parliament's rapporteur for electoral procedure, I am happy that our ideas are now taken forward in this publication.

THE EUROPEAN PARLIAMENT

The European Parliament presents unusual challenges both to the scientist and practitioner. It is one chamber of the legislature of the European Union with a lot of power but little recognition. It reflects a giant historical compromise between the international law principle of the equality of states and the democratic motto of "One person, one vote".

Proportional representation at the EU level needs to bear in mind not only party but also nationality. The European Parliament is the forum of the political single market where the different political cultures and constitutional practices of the 28 member states meet up. MEPs are constitutionally *representatives of the Union's citizens* but they are elected not by a uniform electoral procedure but by different procedures under which separate national political parties and candidates fight it out, largely untroubled by their formal affiliation to European political parties.[1] Efforts to make more uniform the election of the world's first multi-national parliament to be directly elected by universal suffrage have been frustrated.

Voter turnout, as we know, has declined at each election to the European Parliament from 62 percent in 1979 to 43 percent in 2009, although these overall figures disguise sharp contrasts among the states and between elections. The long financial and economic crisis since 2008 has brought to a head a crisis of legitimacy for the European Parliament. If the euro is to be salvaged, and the EU as a whole is to emerge strengthened from its time of trial, transnational democracy needs to work better. Banking union and fiscal union need the installation of federal government. That federal government must be fully accountable to a parliament which connects directly to the citizen and with which the citizen identifies. That parliament must be composed in a fair and logical way best achieved in accordance with a settled arithmetical formula and not as a result of unseemly political bartering which borders on gerrymandering and sparks controversy.

It is probable that in spring 2015 there will be a new round of EU constitutional change. This will take the form of a Convention in which heads of government and the European Commission will talk things through with members of the European and national parliaments. Part of the complex negotiations must include the electoral reform of the European Parliament. This will be the chance to progress CamCom for the apportionment of state seats alongside an ambitious proposal for the creation of a

[1] Article 14(2), Treaty on European Union.

pan-European constituency for which a certain number of MEPs will be elected from transnational party lists.[2]

There is no reason to doubt that the notion of *degressive proportionality*, which strikes mathematicians as odd, will survive these negotiations because it expresses quite well the broadly understood belief that in a federal polity the smaller need to be protected from subordination to the larger. CamCom copes logically with degressive proportionality in a way which should satisfy even the austere requirements of the Bundesverfassungsgericht at Karlsruhe.

Nevertheless, as Friedrich Pukelsheim recognises, fully-fledged CamCom means radical adjustments to the number of MEPs elected in several states. It is important, therefore, that changes to the electoral system for one chamber of the legislature are balanced by changes to the electoral system in the other. Here the Jagiellonian Compromise, which uses the square root as the basis for weighing the votes of the member states in the Council, deserves a good hearing.

In June 2013 the Council and European Parliament eventually agreed that the new Member State of Croatia should have 11 MEPs in the Parliament which is to be elected in May 2014. We worked hard to ensure that the re-apportionment of seats would not contradict the logic of CamCom. There is a first, albeit clumsy, legal definition of degressive proportionality. More importantly, the European Union has now formally decided to pursue the objective of a formulaic approach to the future distribution of seats in the Parliament, coupled with a commitment to revisit the matter of qualified majority voting (QMV) in the Council.

The decision of the European Council, now agreed by the European Parliament, lays down that a new system will be agreed in good time before the 2019 elections which *in future will make it possible, before each fresh election to the European Parliament, to allocate the seats between Member States in an objective, fair, durable and transparent way, translating the principle of degressive proportionality as laid down in Article 1, taking account of any change in their number and demographic trends in their population, as duly ascertained thus respecting the overall balance of the institutional system as laid down in the Treaties.*

So perhaps CamCom and JagCom are destined to surface together in the next EU treaty. Legislators who care to understand the maths should start with this book.

Andrew Duff MEP

Cambridge, United Kingdom
September 2013

[2] For a full exposition of this proposal see Spinelli Group, *A Fundamental Law of the European Union,* Bertelsmann Stiftung 2013.

Preface

Proportional representation systems determine how the political views of individual citizens, who are many, mandate the Members of Parliament, who are but a few. The same techniques apply when in Parliament the political groups are to be represented in a committee of a size much smaller than Parliament itself. There are many similar examples all showing that proportional representation inevitably culminates in the task of translating numbers into numbers—large numbers of those to be represented into small numbers of those serving as representatives. The task is solved by procedures called apportionment methods. Apportionment methods and their applications are the theme of this work. A more detailed *Outline of the Book* follows the *Table of Contents*.

By profession a mathematician rather than a politician, I have had the privilege of getting involved in several proportional representation reform projects in recent years. These include the introduction of a double-proportional electoral system in several Swiss cantons since 2006, the amendment of the German Federal Election Law during 2008–2013, and the discussion of the future composition of the European Parliament. The practical challenges and the teaching experience of many lectures and seminars on the subject of proportional representation and apportionment methods have shaped my view and provided the basis for this book.

Apportionment methods may become quite complex. However, these complexities are no ends in themselves. They are reflections of the historical past of a society, its constitutional framework, its political culture, its identity. On occasion the complexities are due to partisan interests of the legislators responsible. This mélange turns the topic into a truly interdisciplinary project. It draws on such fields as constitutional law, European law, political sciences, medieval history, modern history, discrete mathematics, stochastics, computational algorithms, to name but a few. I became increasingly fascinated by the interaction of so many disciplines. My fascination grew when I had the pleasure of conducting student seminars jointly with colleagues from the humanities on topics of common interest. These experiences made me realize that proportional representation and apportionment methods are a wonderful example to illustrate the *universitas litterarum*, the unity of arts and sciences.

In retrospect I find it much easier to conduct an interdisciplinary seminar than to author an interdisciplinary textbook. Nevertheless I hope that the present book may prove a useful reference work for apportionment methods, for scholars of constitutional law and political sciences as well as for other electoral system designers. The many apportionment methods studied span a wide range of alternatives in Germany, the European Union, and elsewhere. The book not only describes the mechanics of each method, but also lists the method's properties: biasedness in favor of stronger parties at the expense of weaker parties, preferential treatments of groups of stronger parties at the expense of groups of weaker parties, optimality with respect to goodness-of-fit or stability criteria, reasonable dependence on such variables as house size, vote ratios, size of the party system, and so on. These properties are rigorously proved and, whenever possible, substantiated by appropriate formulae.

Since the text developed from notes that I compiled for lectures and seminars, I am rather confident that it can be utilized for these purposes. The material certainly suffices for a lecture course or a student seminar in a curriculum of mathematics, quantitative economics, computational social choice, or electoral system design in the political sciences. I have used parts of the text with particular success in classes for students who are going to be high-school teachers. The chapters presuppose readers with an appreciation for rigorous derivations, and with a readiness to accept arguments from scientific fields other than their own. Most chapters can then be mastered with a minimum knowledge of basic arithmetic. Three chapters involve more technically advanced approaches. Chapters 6 and 7 use some stochastic reasoning, and Chapter 14 discrete optimization and computer algorithms.

The subject of the book is restricted to the quantitative and procedural rules that must be employed when a proportional representation system is implemented; as a consequence the book does *not* explicate the qualitative and normative foundations that would be called for when developing a comprehensive theory of proportional representation. As in all sciences, the classification of quantitative procedures starts with basic methods that later get modified to allow for more ambitious settings. The basic issue is to calculate seat numbers proportionately to vote counts. This task is resolved by divisor methods or by quota methods. Later, geographical subdivisions of the electoral region come into play, as do guarantees for small units to obtain representation no matter how small they are, as do restrictions for stronger groups to limit their representation lest they unduly dominate their weaker partners. In order to respond to these requirements the basic methods are modified into variants that may achieve an impressive degree of complexity.

When teaching the topic I soon became convinced that its intricacies can be appreciated only by contemplating real data. That is, data from actual elections in the real world, rather than imaginary data from contrived elections in the academic ivory tower. My Augsburg students responded enthusiastically and set out to devise an appropriate piece of software, BAZI. BAZI has grown considerably since 2000, and has proved an indispensable tool for carrying out practical calculations and theoretical investigations. I would like to encourage readers of this book to use the program to retrace the examples and to form their own judgment. BAZI is freely available from the website www.uni-augsburg.de/bazi.

ACKNOWLEDGMENTS

My introduction to the proportional representation problem was the monograph of *Michel Balinski / Peyton Young* (1982). In their book the authors recount the apportionment history in the House of Representatives of the United States of America, and then proceed to establish a Theory of Apportionment. This seminal source was soon complemented by the treatise of *Klaus Kopfermann* (1991) who adds the European dimension to the proportional representation heritage. *Svante Janson's* (2012) typescript proved invaluable for specific mathematical questions. These books provide the foundations on which the results of the present work are based.

Several colleagues and friends read parts or all of initial drafts of this book and proposed improvements. I have benefited tremendously from the critical comments and helpful suggestions of *Paul Campbell*, *Rudy Fara*, *Martin Fehndrich*, *Dan Felsenthal*, *Svante Janson*, *Jan Lanke*, and *Daniel Lübbert*.

Throughout the project I had the privilege to rely on the advice and inspiration of my colleagues *Karl Heinz Borgwardt*, *Lothar Heinrich*, and *Antony Unwin* in the Augsburg University Institute for Mathematics. A special word of thanks is due to my non-mathematical Augsburg colleagues who helped me mastering the interdisciplinary aspects of the topic. I wish to thank *Günter Hägele* (Medieval History, University Library), *Thomas Krüger* (Medieval History), *Johannes Masing* (Constitutional Law, now with the University of Freiburg im Breisgau), *Matthias Rossi* (Constitutional Law), and *Rainer-Olaf Schultze* (Political Sciences).

The largest debt of gratitude is due to the current and former members of my workgroup at Augsburg University. Many of them contributed substantially to this work through their research work and PhD theses. Moreover they helped in organizing lectures and seminars, in sorting the material, in optimizing the terminology, in polishing the presentation. For their cooperation I am extremely grateful to *Olga Birkmeier*, *Johanna Fleckenstein*, *Christoph Gietl*, *Max Happacher*, *Thomas Klein*, *Sebastian Maier*, *Kai-Friederike Oelbermann*, *Fabian Reffel*, and *Gerlinde Wolsleben*.

I would like to thank *Andrew Duff* MEP for graciously consenting to contribute the foreword to this book. Finally I wish to acknowledge support of the Deutsche Forschungsgemeinschaft.

FRIEDRICH PUKELSHEIM

Augsburg, Germany
October 2013

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Outline of the Book

CHAPTERS 1, 2: APPORTIONMENT METHODS IN PRACTICE

The initial chapters present an abundance of apportionment methods used in practice. Chapter 1 reviews the European Parliament elections 2009. They provide a rich source of empirical examples. Chapter 2 deals with the German Bundestag election 2009. Emphasis is on the interplay between procedural steps and constitutional requirements. The presentation introduces vital concepts of proportional representation systems beyond their use in European or German elections. Concepts and terminology introduced in these chapters set the scene for the methodological approach that follows.

CHAPTERS 3, 4, 5: DIVISOR METHODS AND QUOTA METHODS

A rigorous apportionment methodology needs to appeal to rounding functions and rounding rules. They are introduced in Chapter 3. Chapters 4 and 5 discuss the two dominant classes of apportionment methods: divisor methods, and quota methods. Usually the sum of all vote counts is much larger than the number of seats available in a parliamentary body. Therefore, a first step converts vote counts into interim quotients of an appropriate order of magnitude. A second step rounds interim quotients to integers. Divisor methods use a flexible divisor for the first step, and a preordained rounding rule for the second. Quota methods employ a formulaic divisor—the quota—for the first step, and a flexible rounding rule for the second.

CHAPTERS 6, 7, 8: DEVIATIONS FROM PROPORTIONALITY

Many apportionment methods deviate from perfect proportionality in a systematic fashion. Chapters 6 and 7 investigate seat biases, that is, averages of the deviations between actual seat numbers and the ideal share of seats assuming that the vote shares follow the uniform distribution or some absolutely continuous distribution. Chapter 8 offers a deterministic comparison of two apportionment methods under the assumption that the vote counts are fixed. The majorization relation is a partial order among apportionment methods indicating whether one method is more beneficial to groups of stronger parties—and hence more disadvantageous to the complementary group of weaker parties—than the other.

CHAPTERS 9, 10, 11: COHERENCE, OPTIMALITY, SPECIFICS

Proportional representation aims at fairly representing voters in terms of their party preferences. Chapter 9 explores the idea that a fair division should be such that every part of it is fair, too. This requirement is captured by the notion of coherence. Divisor methods are coherent, quota methods are not. Chapter 10 evaluates the deviations between actual seat numbers and ideal shares of seats by means of goodness-of-fit criteria. The optimization of particular criteria is shown to lead to particular apportionment methods. Chapter 11 reverses the role of input and output. Given the seat numbers, the range of vote shares is determined that leads to the prespecified number of seats. As a matter of fact it may happen that a straight majority of votes fails to lead to a straight majority of seats. For this reason many electoral laws include an extra majority preservation clause. Three majority clauses are discussed, and their practical usage is illustrated by example.

CHAPTERS 12, 13: PRACTICAL IMPLEMENTATIONS

Many proportional representation systems go beyond abstract proportionality by imposing concrete restrictions. Chapter 12 shows how to handle minimum-maximum restrictions, and empirical examples illustrate their relevance. The most prominent example is the composition of the European Parliament, that is, the allocation of the seats of the European Parliament between the Member States of the Union. Chapter 13 describes the 2013 amendment of the German Federal Election Law. The law achieves impeccable proportionality by adjusting the Bundestag size beyond the nominal level of 598 seats. The system realizes practical equality of the success values of all voters' votes in the whole country. Mild deviations from proportionality may occur when apportioning the seats of a party to its lists of nominees.

CHAPTER 14: DOUBLE PROPORTIONALITY

Chapter 14 treats double-proportional divisor methods. Double-proportionality aims at a fair representation of the geographical division of the electorate as well as of the political division of the voters. The methods achieve this two-way fairness by apportioning seats to districts proportionately to population figures, and seats to parties proportionately to vote counts. The core is the sub-apportionment of seats to each party in each district in such a way that for every district the seats are summing to the given district magnitude, and for every party the seats are summing to their overall proportionate due. To this end two sets of electoral keys are required, district divisors and party divisors. While it is laborious to determine the electoral keys, their publication makes it rather easy to verify the double-proportional seat apportionment.

Notation

$\lfloor t \rfloor, \llbracket t \rrbracket$	floor function, 45, rule of downward rounding, 46
$\lceil t \rceil, \lceil\!\lceil t \rceil\!\rceil$	ceiling function, 47, rule of upward rounding, 47
$\langle t \rangle, \langle\!\langle t \rangle\!\rangle$	commercial rounding, 47, rule of standard rounding, 48
$[t], \llbracket t \rrbracket$	general rounding function, 49, general rounding rule, 49
$\mathbb{N} = \{0, 1, 2, \ldots\}$	set of natural numbers, 44
$s(n), n \in \mathbb{N}$	signpost sequence $\big($always $s(0) = 0\big)$, 51
$s_r(n) = n - 1 + r$	stationary signpost ($n \geq 1$) with split parameter $r \in [0; 1]$, 52
$\widetilde{s}_p(n)$	power-mean signposts with power parameter $p \in [-\infty; \infty]$, 53
$\ell \in \{2, 3, \ldots\}$	number of parties entering the apportionment calculations, 55
$v = (v_1, \ldots, v_\ell)$	vector of vote weights $v_j \in (0; \infty)$ for parties $j \leq \ell$, 55
$v_+ = v_1 + \cdots + v_\ell$	component sum of the vector $v = (v_1, \ldots, v_\ell)$, 55
$h \in \mathbb{N}$	house size, 55
$\mathbb{N}^\ell(h)$	set of seat vectors $x \in \mathbb{N}^\ell$ with component sum $x_+ = h$, 55
$A(h; v)$	set of seat vectors for house size h and vote vector v, 56
A	apportionment rule, 56, apportionment method, 58
v_+/h	votes-per-seats ratio, 41, also known as Hare-quota, 72
$v_j/(v_+/h) = (v_j/v_+)\, h$	ideal share of seats for party $j \leq \ell$, 42
$n+ = \{n, n+1\}$	upward tie, increment option, 64
$n- = \{n-1, n\}$	downward tie, decrement option, 64
$x \preceq y$	majorization of vectors, 111
$A(h, v) \preceq B(h; v)$	majorization of sets of vectors, 112
$A \prec B$	majorization of apportionment methods, 112
$:=$	definitional equality
$\square$	end-of-proof mark
-ward	suffix of adjectives: the downward rounding etc.
-wards	suffix of adverbs: to round downwards etc.
I'	ambiguous prime: complement of the set I
x'	ambiguous prime: transposed vector (or matrix) x
$\bullet$	multi-purpose eye-catcher in tables